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A lower estimate for $\|e^n\|$ ☆

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ABSTRACT

We show that the distance between e^n and its nearest integer is estimated below by $e^{-cn\log n}$ with $c = 15.727$ for all sufficiently large integer n, which improves the earlier results due to Mahler, Mignotte and Wielonsky. Some basic properties on that distance are also discussed.

1. Introduction and main theorem

Let $\{x\}$ be the fractional part of a real number x and put

$$\|x\| = \min\bigl(\{x\}, 1 - \{x\}\bigr),$$

which denotes the distance between x and $\mathbb{Z}$. For a given $\theta > 1$ it seems to be hard to obtain good information about $\{\theta^n\}$ or $\|\theta^n\|$ in general, whereas we know that $\{x^n\}$ distributes uniformly for almost all $x > 1$. Hardy [3] stated that the problem that in what circumstances can it be true that $\|\lambda\theta^n\| \to 0$ when $n \to \infty$ appears to be one of considerable interest and difficulty. Pisot and Salem [8] made an allusion to the difficulty of uniform distribution problem when $\theta = e$ or $\theta = 3/2$.

When θ is an algebraic number, Corvaja and Zannier [1] showed that

$$\lim_{n\to\infty} \|\theta^n\|^{1/n} = 1$$

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if and only if θ^d is not a Pisot number for any positive integer d. However very little is known when θ is a transcendental number. Recently Dubickas [2] constructed a transcendental number $\theta > 1$ satisfying $\{\theta^n\} < \delta_n$ for infinitely many n's, for a given sequence of positive numbers δ_n with $\delta_n^{1/n} \to 0$ $(n \to \infty)$ no matter how fast the δ_n converges to 0.

We now restrict ourselves to the case $\theta = e$. The first lower bound for $\|e^n\|$ was obtained by Mahler in the form

$$\left\|e^n\right\| \geqslant e^{-cn\log n} \tag{1}$$

for all sufficiently large integer n with $c = 40$ in [5] and subsequently with $c = 33$ in [6]. His method is based on the classical Hermite–Padé approximation of type I to the functions $e^x, e^{2x}, \ldots, e^{kx}$. He also stated in his papers that this estimate is rather weak but it does not seem easy to obtain any substantial improvement. Mignotte [7] gave a lower bound with $c = 17.7$ in (1) using the same method, but later Wielonsky [10] pointed out that Mignotte's proof contained errors and the corrected value is $c = 21.012$. Wielonsky succeeded in yielding a new bound with $c = 19.183$ using Hermite–Padé approximation slightly different from the classical one.

In this paper we improve the previous results as follows:

Theorem 1. *The inequality* (1) *holds with* $c = 15.727$ *for all sufficiently large integer n.*

2. Some properties of $\|e^n\|$

In this section we enumerate some basic properties about $\|e^n\|$. Since e is transcendental, it follows from Pisot's theorem that

$$\sum_{n=1}^{\infty}\left\|e^n\right\|^2 = \infty.$$

Moreover the author's result in [4] implies that

$$\#\left\{n \in \mathbb{N};\ \left\|e^n\right\| \geqslant \frac{c_0}{\sqrt{n}}\right\} = \infty, \tag{2}$$

where c_0 is any positive constant smaller than

$$\exp\left(-\frac{1}{2}\int_0^1 \log\log\left(1+\frac{1}{x}\right)dx - 2\right) = 0.12215\ldots.$$

Of course, these are common properties for all transcendental numbers. We do not have any information about the accumulation points of $\{e^n\}$. Dubickas informed the author that it may be an open problem even to disprove that $\|e^n\| \to 0$ as $n \to \infty$. So it may be an interesting problem to replace the order $1/\sqrt{n}$ in (2) by weaker one.

For any $0 < \xi < 1$ we put

$$E(\xi) = \left\{n \in \mathbb{N};\ \left\|e^n\right\| < \xi^n\right\},$$

which is monotone increasing with respect to ξ. About the set $E(\xi)$ we have the following

Theorem 2. *For any* $\xi < 1/e$ *the density of* $E(\xi)$ *is* 0; *that is,*

$$\lim_{n\to\infty}\frac{1}{n}\#\left\{[1,n] \cap E(\xi)\right\} = 0.$$

Proof. Let m be any positive integer. Since each number $e, e^2, \ldots, e^m$ has irrationality measure 2, for any $\eta > 0$ there exists a positive integer $n_0 = n_0(\eta)$ such that

$$\left| e^k - \frac{p}{q} \right| \geqslant \frac{1}{q^{2+\eta}}$$

holds for all $p \in \mathbb{Z}$, $q \geqslant n_0$ and all $1 \leqslant k \leqslant m$. Put $e^n = L_n + \epsilon_n$ where $|\epsilon_n| < 1/2$ and $L_n \in \mathbb{N}$. L_n is the integer nearest to e^n and $|\epsilon_n| = \|e^n\|$. Since

$$e^k - \frac{L_{n+k}}{L_n} = \frac{L_{n+k} + \epsilon_{n+k}}{L_n + \epsilon_n} - \frac{L_{n+k}}{L_n} = \frac{\epsilon_{n+k} L_n - \epsilon_n L_{n+k}}{L_n(L_n + \epsilon_n)},$$

we have for all sufficiently large n satisfying $L_n > n_0$

$$\frac{1}{L_n^{2+\eta}} \leqslant \left| e^k - \frac{L_{n+k}}{L_n} \right| \leqslant \frac{|\epsilon_{n+k}| L_n + |\epsilon_n| L_{n+k}}{L_n e^n}.$$

We can also assume that $2e^{n+m} + 1 < e^{(1+\eta')n}$ where $\eta' = \eta/(2+\eta)$; thus,

$$2 \max\bigl(|\epsilon_n|, |\epsilon_{n+k}|\bigr) \geqslant \frac{e^n}{L_n^{1+\eta} L_{n+m}} > \frac{2e^n}{(2e^n+1)^{1+\eta}(2e^{n+m}+1)} > \frac{2}{e^{(1+2\eta)n}}.$$

Hence we have either $n \notin E(e^{-1-2\eta})$ or $n + k \notin E(e^{-1-2\eta})$. Since $k \in [1, m]$ is arbitrary, this implies that

$$\limsup_{n\to\infty} \frac{1}{n} \#\bigl\{[1, n] \cap E\bigl(e^{-1-2\eta}\bigr)\bigr\} \leqslant \frac{1}{m}.$$

This completes the proof. $\square$

It seems to be natural to conjecture that there will be some $0 < \xi < 1$ for which the set $E(\xi)$ becomes finite. If such a ξ exists, then $\|e^n\| \geqslant \xi^n$ would hold for sufficiently large integer n, which was called "an open problem of Mahler" by Waldschmidt in [9].

3. Construction of Hermite–Padé approximation

Let k and m_j, $0 \leqslant j \leqslant k$, be any positive integers. The classical Hermite–Padé approximation to the functions $e^z, e^{2z}, \ldots, e^{kz}$ is given by the contour integral

$$R(z) = \frac{1}{2\pi i} \int_C F(\zeta) e^{z\zeta}\, d\zeta, \tag{3}$$

where

$$F(\zeta) = \frac{1}{\zeta^{m_0+1}(\zeta - 1)^{m_1+1} \cdots (\zeta - k)^{m_k+1}}$$

and C is any oriented simple closed curve enclosing all poles of $F(\zeta)$. Plainly the entire function $R(z)$ satisfies $R(z) = O(z^{k+M})$ as $z \to 0$ with

$$M = \sum_{j=0}^{k} m_j.$$

Calculating the residues of (3) we have

$$R(z) = \sum_{j=0}^{k} e^{jz} P_j(z) \tag{4}$$

where $P_j(z)$ is the coefficient of w^{-1} in the Laurent expansion at $w = 0$ of

$$\frac{e^{zw}}{(w+j)^{m_0+1} \cdots w^{m_j+1} \cdots (w+j-k)^{m_k+1}};$$

thus, $P_j(z)$ is a polynomial in z of degree m_j with rational coefficients. More precisely, we have

$$P_j(z) = \frac{1}{D_j} \sum_{\substack{i_0+\cdots+i_k=m_j \\ i_0,\ldots,i_k \geqslant 0}} H_j(i_0, i_1, \ldots, i_k) \frac{z^{i_j}}{i_j!},$$

where $D_j = j^{m_0+1} \cdots 1^{m_{j-1}+1} (-1)^{m_{j+1}+1} \cdots (j-k)^{m_k+1}$ and

$$H_j(i_0, i_1, \ldots, i_k) = \prod_{\substack{\ell \neq j \\ 0 \leqslant \ell \leqslant k}} \binom{i_\ell + m_\ell}{i_\ell} \frac{1}{(\ell - j)^{i_\ell}}.$$

Using this formula it is not difficult to see that

$$P_j(z) = \frac{1}{m_0! \cdots m_k! D_j} \int_0^\infty \cdots \int_0^\infty \left(z + \sum_{\substack{\ell \neq j \\ 0 \leqslant \ell \leqslant k}} \frac{x_\ell}{\ell - j} \right)^{m_j} \prod_{\substack{\ell \neq j \\ 0 \leqslant \ell \leqslant k}} \left(e^{-x_\ell} x_\ell^{m_\ell} \, dx_\ell \right).$$

From this real multiple integral expression for $P_j(z)$ we have immediately

$$A_j P_j(z) \in \mathbb{Z}[\![z]\!],$$

where

$$A_j = m_j! |D_j| \Delta_{\max(j,k-j)}^{m_j} \tag{5}$$

and Δ_s is the least common multiple of $1, 2, \ldots, s$, because $\int_0^\infty e^{-x} x^{m+t} \, dx = (m+t)!$ is a multiple of $m!$ for any integer $t \geqslant 0$. Let A be any positive common multiple of $A_0, \ldots, A_k$ and define

$$Q(x, y) = A \sum_{j=0}^{k} P_j(x) y^j \in \mathbb{Z}[\![x, y]\!].$$

Let L_n be the nearest integer to e^n for any positive integer n and put $\epsilon_n = e^n - L_n$, as in the previous section. The following lemma shows how we can obtain a lower estimate for $\|e^n\|$.

Lemma 3. *Suppose that $|Q(n, e^n)| < 1/2$ and $Q(n, L_n) \neq 0$. If $k \leqslant n$, then we have*

$$\|e^n\| > \frac{1}{A T_n} \quad \textit{where } T_n = \max_{1 \leqslant j \leqslant k} e^{jn} |P_j(n)|.$$

Proof. By the mean value theorem there exists some ξ_n lying in the interval between e^n and L_n such that

$$Q\left(n, e^n\right) - Q(n, L_n) = \epsilon_n Q_y(n, \xi_n)$$

where $Q_y(x, y)$ is the partial derivative of $Q(x, y)$ with respect to y. On the other hand, using $\xi_n < \max(e^n, L_n) < e^n + 1/2 < e^{n+1}/2$ and $k \leqslant n$, it holds that

$$\left|Q_y(n, \xi_n)\right| \leqslant A \sum_{j=1}^{k} j\xi_n^{j-1}\left|P_j(n)\right| < \frac{AT_n}{e^n} \sum_{j=1}^{k} j\left(\frac{e}{2}\right)^{j-1} < \frac{AT_n}{2}.$$

Therefore

$$|\epsilon_n| = \frac{|Q(n, e^n) - Q(n, L_n)|}{|Q_y(n, \xi_n)|} > \frac{2|Q(n, L_n)| - 1}{AT_n} \geqslant \frac{1}{AT_n},$$

because $0 \neq Q(n, L_n) \in \mathbb{Z}$. □

The condition $Q(n, L_n) \neq 0$ is essential in the proof of Lemma 3, however it seems to be hard to show this for $Q(x, y)$ directly. This is the reason why we consider the higher derivative of $R(z)$.

For any integer $0 \leqslant d \leqslant k$ the d-th derivative $R^{(d)}(z)$ satisfies that $R^{(d)}(z) = O(z^{k+M-d})$ as $z \to 0$. Indeed it follows immediately from (3) that $R^{(d)}(z)$ corresponds to the Hermite–Padé approximation with the same parameters as $R(z)$ except for m_0; that is, one gets $R^{(d)}(z)$ by taking $m_0 - d$ instead of m_0 when $m_0 \geqslant d$. On the other hand, it is easily seen from (4) that

$$R^{(d)}(z) = \sum_{j=0}^{k} e^{jz} P_j^{[d]}(z),$$

where

$$P_j^{[d]}(z) = \left(\frac{d}{dz} + j\,\mathrm{Id}\right)^d P_j(z) \tag{6}$$

and Id is the identity operator. Note that $\deg P_0^{[d]} = m_0 - d$ when $m_0 \geqslant d$ and $\deg P_j^{[d]} = m_j$ for $1 \leqslant j \leqslant k$. Finally put $Q^{[d]}(x, y) = \sum_{j=0}^{k} P_j^{[d]}(x) y^j$, where we do not take care of an integral factor.

Lemma 4. *For any positive integers k, n and $m_0, m_1, \ldots, m_k$ there exists at least one $d \in [0, k]$ satisfying $Q^{[d]}(n, L_n) \neq 0$.*

Proof. Suppose, on the contrary, that $Q^{[\ell]}(n, L_n) = 0$ for all $0 \leqslant \ell \leqslant k$. This means that the system of linear homogeneous equations:

$$\sum_{j=0}^{k} P_j^{[\ell]}(n) X_j = 0 \quad (0 \leqslant \ell \leqslant k)$$

possesses a non-trivial solution $(X_0, X_1, \ldots, X_k) = (1, L_n, \ldots, L_n^k)$; hence $\Phi(n) = 0$ where

$$\Phi(z) = \det\left(P_j^{[\ell]}(z)\right)_{0 \leqslant \ell, j \leqslant k}.$$

$\Phi(z)$ is a polynomial of degree less than or equal to $m_0+\cdots+m_k=M$. Moreover, adding j-th column multiplied by e^{jz} to the first column for $1\leqslant j\leqslant k$, one gets a new matrix whose first column is

$$^t\big(R(z), R'(z), \ldots, R^{(k)}(z)\big);$$

thus, we have $\Phi(z)=O(z^M)$. This implies that $\Phi(z)=c_1z^M$ for some constant c_1, which is equal to

$$a_0 \det\big(j^\ell a_j\big)_{1\leqslant j,\ell\leqslant k}$$

from (6), where $a_j\neq 0$ is the leading coefficient of $P_j(z)$ for $0\leqslant j\leqslant k$. We thus have $c_1\neq 0$ by using Vandermonde determinant; therefore $\Phi(n)\neq 0$ and this contradiction completes the proof. □

Let $f:[0,1]\to[0,1]$ be a Lipschitz continuous function; that is, there exists a positive constant K such that $|f(x)-f(y)|\leqslant K|x-y|$ for any $0\leqslant x,y\leqslant 1$. We further assume that $\max_{0\leqslant x\leqslant 1} f(x)=1$ and put

$$\alpha=\int_0^1 f(x)\,dx\in(0,1].$$

Suitable piecewise linear functions, so-called "zig-zag" functions, give typical such examples. Lipschitz continuous functions have a nice property about the difference between the integral and its Riemann sum; namely,

$$\int_0^1 f(x)\,dx-\frac{1}{k}\sum_{j=0}^{k} f\left(\frac{j}{k}\right)=O\left(\frac{1}{k}\right)\quad\text{as } k\to\infty. \tag{7}$$

For any positive integer m we then put

$$m_j=\left[f\left(\frac{j}{k}\right)m\right]\quad\text{for } 0\leqslant j\leqslant k;$$

thus, the parameters $m_0,\ldots,m_k$ are controlled by m and the function f. The ordinary Hermite–Padé approximations used by Mahler [5,6] and Mignotte [7] correspond to the case $f(x)=1$. On the other hand, the approximation used by Wielonsky [10] may correspond to the step function

$$f(x)=\begin{cases}1/51 & \text{for } 0\leqslant x\leqslant 0.07 \text{ and } 0.93\leqslant x\leqslant 1,\\ 1 & \text{otherwise,}\end{cases}$$

being a small perturbation from the ordinary case. Of course, many results described later are valid for discontinuous piecewise C^1 functions like the above Wielonsky' case, but the continuity of f makes our arguments rather simpler.

4. Estimates for remainder term and residues

We first give an upper estimate for $R^{(d)}(n)=Q^{[d]}(n,e^n)$.

Lemma 5. *For arbitrarily fixed $\rho > 1$ we put*

$$\kappa = \int_0^1 \frac{f(x)}{\rho - x}\,dx.$$

Then, for any positive integers k, m, n satisfying $n \leqslant \kappa m$ and $k \leqslant m$, we have

$$\left|Q^{[d]}\big(n, e^n\big)\right| \leqslant \exp\big(m\big(-\alpha k \log k - \beta k + O(\log k)\big)\big)$$

as $k \to \infty$, where

$$\beta = \int_0^1 \left(\log(\rho - x) - \frac{\rho}{\rho - x}\right) f(x)\,dx$$

and $O(\log k)$ is independent of m, n and $d \in [0, k]$.

Proof. We take the circle $|\zeta| = \rho k$ as the contour C in (3). Noticing that $|e^{n\zeta}| = e^{n\Re\zeta} \leqslant e^{n\rho k} \leqslant (e^{\kappa\rho k})^m$ and

$$|\zeta - j|^{m_j+1} \geqslant (\rho k - j)^{m_j+1} > (\rho k - j)^{f(j/k)m}$$

for $0 \leqslant j \leqslant k$ when $k \geqslant 1/(\rho - 1)$, it follows from (3) that

$$\left|R^{(d)}(n)\right| \leqslant e^{Gm} \times \frac{1}{2\pi} \int_{|\zeta|=\rho k} |\zeta|^d\,|d\zeta| \leqslant \exp\big(Gm + (k+1)\log(\rho k)\big)$$

where

$$G = \kappa\rho k - \sum_{j=0}^{k} f\left(\frac{j}{k}\right) \log(\rho k - j).$$

Clearly $(k+1)\log(\rho k) = m O(\log k)$ and from (7)

$$\begin{aligned} G &= -k \log k \cdot \frac{1}{k} \sum_{j=0}^{k} f\left(\frac{j}{k}\right) + k\left(\kappa\rho - \frac{1}{k} \sum_{j=0}^{k} f\left(\frac{j}{k}\right) \log\left(\rho - \frac{j}{k}\right)\right) \\ &= -\alpha k \log k - \beta k + O(\log k). \end{aligned}$$

This completes the proof. □

Note that β is the maximum of the function $U(u)$ in the range $u > 1$, which is attained uniquely at $u = \rho$, where

$$U(u) = \int_0^1 f(x) \log(u - x)\,dx - \kappa u,$$

because $U(u)$ is a concave function on $(1, \infty)$.

Next, for $0 \leqslant d \leqslant k$, we put

$$T_n^{[d]} = \max_{1 \leqslant j \leqslant k} e^{jn} \big| P_j^{[d]}(n) \big|.$$

We have the following upper estimate for $T_n^{[d]}$ similar to the previous lemma.

Lemma 6. *For any positive integers k, m, n satisfying $n \leqslant \kappa m$ and $k \leqslant m$ we have*

$$T_n^{[d]} \leqslant \exp\big(m\big(-\alpha k \log k + \delta k + O(\log k)\big)\big)$$

as $k \to \infty$, where δ is the maximum of the function

$$V(u) = \kappa u - \int_0^1 f(x) \log|u - x|\, dx$$

in the range $0 \leqslant u \leqslant 1$ and $O(\log k)$ is independent of m, n and $d \in [0, k]$.

Proof. Since $e^{jn} P_j^{[d]}(n)$ is the residue of $F(\zeta)e^{n\zeta}$ at $\zeta = j$ with $m_0' = m_0 - d$ instead of m_0, this is equal to the integral (3) if we take the circle centered at $\zeta = j$ with radius $1/2$ as the contour C. We then have $|e^{n\zeta}| \leqslant e^{\kappa m(j+1/2)}$ and

$$|\zeta - \ell|^{m_\ell + 1} \geqslant \begin{cases} 2^{-f(\ell/k)m-1} & \text{if } \ell = j, j \pm 1, \\ \min(|j - \ell + 1/2|, |j - \ell - 1/2|)^{f(\ell/k)m} & \text{otherwise.} \end{cases}$$

Hence, for $1 \leqslant j \leqslant k$,

$$\begin{aligned} e^{jn} \big| P_j^{[d]}(n) \big| &\leqslant 4e^{G_j m} \times \frac{1}{2\pi} \int_{|\zeta - j| = 1/2} |\zeta|^d \, |d\zeta| \\ &\leqslant \exp\left(G_j m + k \log\left(k + \frac{1}{2}\right) + \frac{\kappa m}{2} + (m+2)\log 2\right) \\ &= \exp\big(G_j m + m O(\log k)\big), \end{aligned}$$

where

$$\begin{aligned} G_j &= \kappa j - \sum_{\ell=0}^{j-1} f\left(\frac{\ell}{k}\right) \log\left(j - \ell - \frac{1}{2}\right) - \sum_{\ell=j+1}^{k} f\left(\frac{\ell}{k}\right) \log\left(\ell - j - \frac{1}{2}\right) \\ &= -k \log k \cdot \frac{1}{k} \sum_{\ell=0}^{k} f\left(\frac{\ell}{k}\right) + k\left(\frac{\kappa j}{k} - \frac{1}{k} \sum_{\ell=0}^{j-1} f\left(\frac{\ell}{k}\right) \log\left(\frac{j}{k} - \frac{\ell}{k} - \frac{1}{2k}\right) \right. \\ &\quad \left. - \frac{1}{k} \sum_{\ell=j+1}^{k} f\left(\frac{\ell}{k}\right) \log\left(\frac{\ell}{k} - \frac{j}{k} - \frac{1}{2k}\right)\right) + O(\log k). \end{aligned}$$

Since $\log x$ is absolutely integrable, it is not difficult to see that the second and third sums in the last expression can be estimated by the improper integrals:

$$-\int_0^{j/k} f(x)\log\left(\frac{j}{k}-x\right)dx \quad \text{and} \quad -\int_{j/k}^1 f(x)\log\left(x-\frac{j}{k}\right)dx$$

respectively within an error $O((\log k)/k)$. Therefore

$$G_j = -\alpha k\log k + V\left(\frac{j}{k}\right)k + O(\log k)$$

and

$$T_n^{[d]} = \max_{1\leqslant j\leqslant k} e^{jn}\left|P_j^{[d]}(n)\right| \leqslant \exp\bigl(m\bigl(-\alpha k\log k + \delta k + O(\log k)\bigr)\bigr),$$

as required. Note that $V(u)$ is a continuous function on $[0,1]$. □

5. Some basic tools

This section is devoted to prepare some basic tools for the next section, which is a crucial part of this paper. Let $f(x)$ be the function introduced in Section 3. We denote by $\tilde{f}(x)$ its zero extension to $\mathbb{R}$; that is,

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } 0\leqslant x\leqslant 1,\\ 0 & \text{otherwise.}\end{cases}$$

For any $\tau \in \mathbb{R}$ and $\omega > 0$ we define

$$\Lambda(\tau,\omega) = \sum_{\ell\in\mathbb{Z}} \tilde{f}(\tau\omega + \ell\omega). \tag{8}$$

Note that $\Lambda(\tau,\omega)$ is a periodic function with period 1 in τ for each ω and that the number of non-zero terms in the sum (8) is at most $[1/\omega]+1$.

Moreover, as the function in ω, $\Lambda(\tau,\omega)$ behaves almost like α/ω, as follows.

Lemma 7. *There exists a positive constant K_0 independent of $\tau \in \mathbb{R}$ and $\omega > 0$ such that*

$$\left|\Lambda(\tau,\omega) - \frac{\alpha}{\omega}\right| \leqslant K_0.$$

Proof. We can assume that $0\leqslant \tau < 1$ and $0 < \omega < 1$. Since $\tau\omega - \omega < 0$ and $\tau\omega + ([1/\omega]+1)\omega > 1$, ℓ varies from 0 to $[1/\omega]$ in (8). We put

$$I_\ell = \frac{1}{\omega}\int_{\tau\omega+\ell\omega}^{\tau\omega+(\ell+1)\omega} \tilde{f}(x)\,dx - \tilde{f}(\tau\omega+\ell\omega)$$

for $-1 \leqslant \ell \leqslant [1/\omega]$. Then it follows from Lipschitz continuity that

$$|I_\ell| \leqslant \frac{1}{\omega} \int_{\tau\omega+\ell\omega}^{\tau\omega+(\ell+1)\omega} \bigl|f(x) - f(\tau\omega+\ell\omega)\bigr|\,dx \leqslant \frac{K\omega}{2}$$

for $0 \leqslant \ell < [1/\omega]$. Since $|I_{-1}| \leqslant 1$ and $|I_{[1/\omega]}| \leqslant 1$, we have

$$\left|\frac{1}{\omega}\int_0^1 f(x)\,dx - \Lambda(\tau,\omega)\right| \leqslant \sum_{\ell=-1}^{[1/\omega]} |I_\ell| \leqslant 2 + \frac{K\omega}{2}\left[\frac{1}{\omega}\right] \leqslant 2 + \frac{K}{2}. \qquad \square$$

Note that α/ω is equal to the average of Λ over one period, because

$$\int_0^1 \Lambda(\tau,\omega)\,d\tau = \sum_{\ell=0}^{[1/\omega]} \int_0^1 \tilde{f}(\tau\omega+\ell\omega)\,d\tau = \frac{1}{\omega}\sum_{\ell=0}^{[1/\omega]} \int_{\ell\omega}^{(\ell+1)\omega} \tilde{f}(x)\,dx = \frac{1}{\omega}\int_0^1 f(x)\,dx.$$

Lemma 7 implies that the function

$$\lambda(\tau,\omega) = \Lambda(\tau,\omega) - \alpha\left(\left[\frac{1}{\omega}\right] + 1\right)$$

is bounded on $\mathbb{R} \times \mathbb{R}_+$, which we call the "finite part" of Λ.

Hereafter we assume that $f(0) = f(1) = 0$ in addition, which implies $\tilde{f} \in \mathrm{C}(\mathbb{R})$, although this condition excludes the ordinary case $f(x) = 1$. For any compact subset $E \subset \mathbb{R} \times \mathbb{R}_+$ it is clear that there exists a positive integer ℓ_0 satisfying $\tau\omega + \ell\omega \notin [0,1]$ for any $|\ell| \geqslant \ell_0$ and $(\tau,\omega) \in E$. This means that the series in (8) converges uniformly on E; therefore $\Lambda \in \mathrm{C}(\mathbb{R} \times \mathbb{R}_+)$.

For $\omega > 0$ we put

$$\Psi(\omega) = \max_{0 \leqslant \tau \leqslant 1} \Lambda(\tau,\omega). \tag{9}$$

It follows from the next lemma that $\Psi \in \mathrm{C}(\mathbb{R}_+)$.

Lemma 8. *Suppose that $g \in \mathrm{C}([a,b] \times [c,d])$ and put $h(y) = \max_{a \leqslant x \leqslant b} g(x,y)$ for $c \leqslant y \leqslant d$. Then $h \in \mathrm{C}[c,d]$.*

Proof. Suppose, on the contrary, that $h(y)$ is discontinuous at $y_0 \in [c,d]$. Then there exist a sequence $y_n \in [c,d]$ and a constant $\eta > 0$ such that $y_n \to y_0$ $(n \to \infty)$ and $|h(y_n) - h(y_0)| \geqslant \eta$. Put $h(y_n) = g(x_n, y_n)$ for $n \geqslant 0$. Without loss of generality, we can assume that the sequence x_n converges to some point $x^* \in [a,b]$. We have $h(y_0) \geqslant g(x^*, y_0) + \eta$.

On the other hand, for sufficiently large n we have

$$\bigl|h(y_n) - g\bigl(x^*, y_0\bigr)\bigr| < \frac{\eta}{2} \quad \text{and} \quad \bigl|g(x_0, y_n) - h(y_0)\bigr| < \frac{\eta}{2}.$$

Thus we obtain

$$h(y_0) \geqslant g\bigl(x^*, y_0\bigr) + \eta > h(y_n) + \frac{\eta}{2} \geqslant g(x_0, y_n) + \frac{\eta}{2} > h(y_0),$$

a contradiction. $\square$

The corresponding maximum of the finite part

$$\psi(\omega) = \Psi(\omega) - \alpha\left(\left[\frac{1}{\omega}\right] + 1\right)$$

is therefore Riemann-integrable on $[0, 1]$, because it is bounded and the set of discontinuity points are countable. We will apply the following lemma to $\psi(\omega)$.

Lemma 9. *Let $\varphi(x)$ be a Riemann-integrable function defined on* $[0, 1]$. *We then have*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{\substack{p\leqslant k \\ p:\text{ prime}}} \varphi\left(\frac{p}{k}\right) \log p = \int_0^1 \varphi(x)\, dx.$$

Proof. It follows from the prime number theorem that the lemma holds for any characteristic function of a subinterval in $[0, 1]$, hence for any step function as a finite linear combination of such functions.

We first consider the case $\varphi \in \mathrm{C}[0, 1]$. For any $\epsilon > 0$ there exists a positive integer r such that

$$\left| \int_0^1 \varphi(x)\, dx - \int_0^1 \varphi_{\text{step}}(x)\, dx \right| < \epsilon$$

and $|\varphi(x) - \varphi_{\text{step}}(x)| < \epsilon$ for any $x \in [0, 1]$ by the uniform continuity, where $\varphi_{\text{step}}(x)$ is the step function defined by $\varphi_{\text{step}}(x) = \varphi(j/r)$ for $j/r \leqslant x < (j + 1)/r$, $j = 0, \ldots, r - 1$ and $\varphi_{\text{step}}(1) = \varphi(1 - 1/r)$. Letting $k \to \infty$ in the inequality

$$\left| \frac{1}{k} \sum_{\substack{p\leqslant k \\ p:\text{ prime}}} \varphi\left(\frac{p}{k}\right) \log p - \frac{1}{k} \sum_{\substack{p\leqslant k \\ p:\text{ prime}}} \varphi_{\text{step}}\left(\frac{p}{k}\right) \log p \right| < \frac{\epsilon}{k} \sum_{\substack{p\leqslant k \\ p:\text{ prime}}} \log p,$$

we have

$$\limsup_{k\to\infty} \left| \frac{1}{k} \sum_{\substack{p\leqslant k \\ p:\text{ prime}}} \varphi\left(\frac{p}{k}\right) \log p - \int_0^1 \varphi(x)\, dx \right| \leqslant \epsilon + \epsilon = 2\epsilon.$$

Now let φ be any Riemann-integrable function on $[0, 1]$. It can be seen that, for any $\epsilon > 0$, there exist $\varphi^{\pm} \in \mathrm{C}[0, 1]$ such that $\varphi^{-}(x) \leqslant \varphi(x) \leqslant \varphi^{+}(x)$ and

$$\left| \int_0^1 \varphi(x)\, dx - \int_0^1 \varphi^{\pm}(x)\, dx \right| < \epsilon.$$

Letting $k \to \infty$ in the inequalities

$$\frac{1}{k} \sum_{\substack{p\leqslant k \\ p:\text{ prime}}} \varphi^{-}\left(\frac{p}{k}\right) \log p \leqslant \frac{1}{k} \sum_{\substack{p\leqslant k \\ p:\text{ prime}}} \varphi\left(\frac{p}{k}\right) \log p \leqslant \frac{1}{k} \sum_{\substack{p\leqslant k \\ p:\text{ prime}}} \varphi^{+}\left(\frac{p}{k}\right) \log p,$$

we have

$$\limsup_{k\to\infty}\left|\frac{1}{k}\sum_{\substack{p\leqslant k\\ p:\ \text{prime}}}\varphi\left(\frac{p}{k}\right)\log p-\int_0^1\varphi(x)\,dx\right|\leqslant\epsilon,$$

which completes the proof. □

6. Estimate for integral factor A

Our aim is to construct a positive integer $A=A(k,m)$ satisfying $AP_j^{[d]}(n)\in\mathbb{Z}$ for all $0\leqslant j\leqslant k$ and $0\leqslant d\leqslant k$ having an upper estimate in the form

$$A\leqslant\exp\bigl(m\bigl(\log m+\alpha k\log k+\sigma k+o(k)\bigr)\bigr)$$

as $k\to\infty$ with some constant σ. We will see later that k should be taken as $O(\log m)$ so that the principal part $\alpha mk\log k$ in the exponents of Lemmas 5 and 6 will be canceled out after multiplying the integral factor A.

Since $m_j\leqslant m$ and $m_0'=m_0-d\leqslant m_0$, it follows from (5) that the integer A can be obtained by $A=m!k!B$ where B is some positive common multiple of

$$B_j=j^{m_0}\cdots 1^{m_{j-1}}\cdot 1^{m_{j+1}}\cdots(k-j)^{m_k}\Delta_{\max(j,k-j)}^{m_j}$$

for $0\leqslant j\leqslant k$. Factorizing B_j into prime factors we can write

$$B_j=\prod_{\substack{p\leqslant k\\ p:\ \text{prime}}}p^{\nu_j(p)},$$

where

$$\nu_j(p)=\sum_{\ell,s}{}^{*}m_{j+\ell p^s}+m_j\mu_j(p),$$
$$\mu_j(p)=\left[\frac{\log\max(j,k-j)}{\log p}\right]$$

and the sum $\sum_{\ell,s}^{*}$ extends over all integers $\ell\neq 0$ and $s\geqslant 1$ satisfying $j+\ell p^s\in[0,k]$. Using $m_j\leqslant f(j/k)m$ and (8) we thus have

$$\nu_j(p)\leqslant m\left(\sum_{s\geqslant 1}\left(\Lambda\left(\frac{j}{p^s},\frac{p^s}{k}\right)-f\left(\frac{j}{k}\right)\right)+\mu_j(p)f\left(\frac{j}{k}\right)\right).\tag{10}$$

Note that s runs up to $[\log k/\log p]$ in the right-hand side. To estimate the maximum of $\nu_j(p)$ as j varies we must distinguish three cases, as follows.

Lemma 10. *Suppose that $k\geqslant 4$ and p is any prime number in the interval $(k/2,k]$. Putting $\omega=p/k\in(1/2,1]$ we have*

$$\max_{0\leqslant j\leqslant k}\nu_j(p)\leqslant m\max_{0\leqslant x\leqslant 1-\omega}\bigl(f(x)+f(x+\omega)\bigr).$$

Proof. Since $\sqrt{k} < p$, the sum in s in the right-hand side of (10) consists only of a single term corresponding to $s = 1$. Hence, putting $\tau = j/p$,

$$\nu_j(p) \leqslant m\big(\Lambda(\tau,\omega) - f(\tau\omega) + \mu_j(p) f(\tau\omega)\big).$$

Here $\mu_j(p)$ is either 0 or 1 according as $j \in (k-p, p)$ or $j \in [0, k-p] \cup [p, k]$ respectively. These cases correspond to $\tau \in (1/\omega - 1, 1)$ or $\tau \in [0, 1/\omega - 1] \cup [1, 1/\omega]$ respectively. In the former case $\tau\omega + \ell\omega \in [0,1]$ occurs if and only if $\ell = 0$; hence $\Lambda(\tau,\omega) = f(\tau\omega)$ and $\nu_j(p) = 0$. The latter case reduces to $\tau \in [0, 1/\omega - 1]$ by the periodicity of Λ and we then have $\Lambda(\tau,\omega) = f(\tau\omega) + f(\tau\omega + \omega)$, because $2\omega > 1$. Note that $x = \tau\omega \in [0, 1-\omega]$. $\square$

Lemma 11. *Suppose that $k \geqslant 6$ and p is any prime number in the interval $(\sqrt{k}, k/2]$. Putting $\omega = p/k \in (0, 1/2]$ we have*

$$\max_{0 \leqslant j \leqslant k} \nu_j(p) \leqslant m\Psi(\omega).$$

Proof. Since $p \leqslant k/2 \leqslant \max(j, k-j)$ for any j, we have $\mu_j(p) = 1$. In this case the sum in s in the right-hand side of (10) consists also only of a single term. $\square$

Lemma 12. *Suppose that $k \geqslant 4$ and p is any prime number in the interval $[2, \sqrt{k}]$. We have*

$$\max_{0 \leqslant j \leqslant k} \nu_j(p) \leqslant m\left(\alpha \sum_{s \geqslant 1} \left[\frac{k}{p^s}\right] + O\left(\frac{\log k}{\log p}\right)\right)$$

where the constant contained in O-symbol is independent of p and k.

Proof. Since $\mu_j(p) \leqslant [\log k / \log p]$, it follows from (10) and Lemma 7 that

$$\nu_j(p) \leqslant m\left(\sum_{s=1}^{[\log k/\log p]} \Lambda\left(\frac{j}{p^s}, \frac{p^s}{k}\right) + \left[\frac{\log k}{\log p}\right]\right) \leqslant m\left(\alpha \sum_{s \geqslant 1} \left[\frac{k}{p^s}\right] + O\left(\frac{\log k}{\log p}\right)\right).$$

In this case the finite part does not contribute to the estimate for A. $\square$

We finally define $\phi(\omega) = \psi(\omega) - \Psi_0(\omega)$ for $0 < \omega \leqslant 1$ where

$$\Psi_0(\omega) = \begin{cases} 0 & \text{for } 0 < \omega \leqslant 1/2, \\ \Psi(\omega) - \max_{0 \leqslant \tau \leqslant 1/\omega - 1} \Lambda(\tau,\omega) & \text{for } 1/2 < \omega \leqslant 1. \end{cases}$$

It follows from Lemmas 10, 11 and 12 that the desired integer A can be given by $A = m!k!\Gamma_1\Gamma_2\Gamma_3$ where

$$\Gamma_1 = \prod_{\substack{\sqrt{k} < p \leqslant k \\ p:\ \text{prime}}} \exp\left(\left[m\phi\left(\frac{p}{k}\right) + m\alpha\left[\frac{k}{p}\right] + m\alpha\right]\log p\right),$$

$$\Gamma_2 = \prod_{\substack{p \leqslant \sqrt{k} \\ p:\ \text{prime}}} \exp\left(\left[m\alpha \sum_{s \geqslant 1} \left[\frac{k}{p^s}\right]\right]\log p\right),$$

$$\Gamma_3 = \prod_{\substack{p \leqslant \sqrt{k} \\ p:\ \text{prime}}} \exp\left(\left[m O\left(\frac{\log k}{\log p}\right) + 1\right] \log p\right).$$

Clearly Γ_3 is negligible because $\Gamma_3 \leqslant \prod_{p \leqslant \sqrt{k}} e^{mO(\log k)} = e^{mO(\sqrt{k}\log k)} = e^{mo(k)}$. Moreover the product $\Gamma_1 \Gamma_2$ is a divisor of $\Gamma_1' \Gamma_2' \Gamma_3' \Gamma_4'$ where

$$\Gamma_1' = \prod_{\substack{p \leqslant k \\ p:\ \text{prime}}} \exp\left(\sum_{s \geqslant 1} \left[m\alpha \left[\frac{k}{p^s}\right]\right] \log p\right),$$

$$\Gamma_2' = \prod_{\substack{p \leqslant k \\ p:\ \text{prime}}} p^{[m\alpha]},$$

$$\Gamma_3' = \prod_{\substack{p \leqslant k \\ p:\ \text{prime}}} \exp\left(\left[m\phi\left(\frac{p}{k}\right)\right] \log p\right)$$

and

$$\Gamma_4' = \prod_{\substack{p \leqslant k \\ p:\ \text{prime}}} p^2 \times \prod_{\substack{p \leqslant \sqrt{k} \\ p:\ \text{prime}}} p^{[\log k/\log p]} = e^{O(k)}.$$

We then have

$$\Gamma_1' \leqslant \prod_{\substack{p \leqslant k \\ p:\ \text{prime}}} \exp\left(m\alpha \sum_{s \geqslant 1} \left[\frac{k}{p^s}\right] \log p\right) = (k!)^{m\alpha} = \exp\bigl(m\alpha\bigl(k\log k - k + O(\log k)\bigr)\bigr)$$

and

$$\Gamma_2' \leqslant \prod_{\substack{p \leqslant k \\ p:\ \text{prime}}} p^{m\alpha} \leqslant \Delta_k^{m\alpha} = \exp\bigl(m\alpha\bigl(k + o(k)\bigr)\bigr)$$

by the prime number theorem. For Γ_3' it follows from Lemma 9 that

$$\Gamma_3' \leqslant \prod_{\substack{p \leqslant k \\ p:\ \text{prime}}} \exp\left(m\phi\left(\frac{p}{k}\right) \log p\right) = \exp\bigl(m\bigl(\sigma k + o(k)\bigr)\bigr),$$

where

$$\sigma = \int_0^1 \phi(x)\, dx.$$

Therefore we obtain the following

Lemma 13. *For any positive integers k, m satisfying $k \leqslant m$ we have*

$$A \leqslant \exp\bigl(m\bigl(\log m + \alpha k \log k + \sigma k + o(k)\bigr)\bigr)$$

as $k \to \infty$.

7. Proof of Theorem 1

Lemma 3 means that the smaller upper estimate for $AT_n^{[d]}$ derives the better lower estimate for $\|e^n\|$. Since it follows from Lemmas 6 and 13 that

$$\max_{0\leqslant d\leqslant k} AT_n^{[d]} \leqslant \exp\bigl(m\bigl(\log m + (\delta+\sigma)k + o(k)\bigr)\bigr),$$

m should be taken as small as possible; so we set $m = [n/\kappa] + 1$. We also have from Lemmas 5 and 13

$$\max_{0\leqslant d\leqslant k} A\bigl|Q^{[d]}\bigl(n, e^n\bigr)\bigr| \leqslant \exp\bigl(m\bigl(\log m + (\sigma-\beta)k + o(k)\bigr)\bigr),$$

from which we must expect that $\beta > \sigma$ and $\log m$ should have the equivalent order as k. Thus we put $k = [a \log n] + 1$ for some constant $a > 0$ so that

$$\max_{0\leqslant d\leqslant k} AT_n^{[d]} \leqslant \exp\bigl(\bigl(1 + a(\delta+\sigma) + o(1)\bigr)m \log n\bigr)$$

and

$$\max_{0\leqslant d\leqslant k} A\bigl|Q^{[d]}\bigl(n, e^n\bigr)\bigr| \leqslant \exp\bigl(\bigl(1 + a(\sigma-\beta) + o(1)\bigr)m \log n\bigr)$$

as $n \to \infty$. Taking

$$a = \frac{1}{\beta-\sigma} + \epsilon$$

for arbitrarily fixed $\epsilon > 0$ when $\beta > \sigma$, we get from Lemma 3 the following

Theorem 14. *The inequality $\|e^n\| \geqslant e^{-cn\log n}$ holds for all sufficiently large integer n provided that $\beta > \sigma$, where the constant c is any number greater than*

$$\Omega(f) = \frac{\beta+\delta}{\kappa(\beta-\sigma)}.$$

Note that $\Omega(f)$ is a functional on some subset of the unit sphere of the space of Lipschitz continuous functions with maximum norm. However it seems to be hard to analyze the infimum of $\Omega(f)$. We only give a “simple” example of $f(x)$ with $\Omega(f) < 15.727$ for a suitable ρ, which certainly proves our main theorem.

For any $\xi \in (0, 1/2)$ we consider the piecewise linear function defined by

$$f_\xi(x) = \begin{cases} x/\xi & \text{for } 0 \leqslant x \leqslant \xi, \\ 1 & \text{for } \xi < x \leqslant 1-\xi, \\ (1-x)/\xi & \text{for } 1-\xi < x \leqslant 1, \end{cases}$$

which we call a *trapezoidal* function. Obviously it satisfies the conditions for $f(x)$ mentioned in Sections 3 and 5 with $\alpha = 1 - \xi$. Put

$$Z(x) = \int_0^x \int_0^v \log|u|\,du\,dv = \frac{x^2}{2}\log|x| - \frac{3}{4}x^2$$

for $x \in \mathbb{R}$ and define the difference operator Δ_ξ by

$$\Delta_\xi g(x) = \frac{1}{\xi}\big(g(x) - g(x-\xi) - g(x-1+\xi) + g(x-1)\big).$$

Then it can be easily seen that

$$\int_0^1 f_\xi(x)\log|u-x|\,dx = \Delta_\xi Z(u)$$

holds for any $u \in \mathbb{R}$; therefore we obtain

$$\kappa = \int_0^1 \frac{f_\xi(x)}{\rho - x}\,dx = \Delta_\xi Z'(\rho),$$

$$\beta = \int_0^1 f_\xi(x)\log(\rho - x)\,dx - \kappa\rho = \Delta_\xi Z(\rho) - \rho\Delta_\xi Z'(\rho),$$

$$\delta = \max_{0\leqslant u\leqslant 1} W_\xi(u),$$

where $W_\xi(u) = u\Delta_\xi Z'(\rho) - \Delta_\xi Z(u)$. Noticing that

$$W_\xi'''(u) = -\Delta_\xi Z'''(u) = \frac{(1-\xi)(1-2u)}{u(u-\xi)(u-1+\xi)(u-1)},$$

the function $W_\xi(u)$ attains its maximum on $[0,1]$ at a unique point $u^* \in (0,1)$, because

$$W_\xi''\left(\frac{1}{2}\right) = \frac{2}{\xi}\log(1-2\xi) < 0$$

and

$$W_\xi'(1) = \Delta_\xi Z'(\rho) - \Delta_\xi Z'(1) < 0.$$

Note also that $\Delta_\xi Z' \in \mathrm{C}(\mathbb{R})$ is strictly monotone decreasing on $[1,\infty)$.

It may be surprising that we encounter the function $\Delta_\xi Z(u)$ even in the calculation of σ. We need the following two lemmas.

Lemma 15. *If $f(x) = f(1-x)$ holds, then $\Lambda(\tau,\omega) = \Lambda(\{1/\omega\} - \tau, \omega)$.*

Proof.

$$\Lambda(\tau,\omega) = \sum_{\ell\in\mathbb{Z}} \tilde{f}(1-\tau\omega-\ell\omega) = \sum_{\ell\in\mathbb{Z}} \tilde{f}\left(\left(\frac{1}{\omega}-\tau\right)\omega+\ell\omega\right)$$
$$= \Lambda\left(\frac{1}{\omega}-\tau,\omega\right) = \Lambda\left(\left\{\frac{1}{\omega}\right\}-\tau,\omega\right). \qquad \square$$

Lemma 16. *If $f(x)$ is a piecewise linear function and $0=\tau_0<\tau_1<\cdots<\tau_s=1$ are the breakpoints of $\tilde{f}(x)$, then $\Lambda(\tau,\omega)$ is also a piecewise linear function in τ with breakpoints in $[0,1)$ at most*

$$0, \left\{\frac{\tau_1}{\omega}\right\}, \ldots, \left\{\frac{\tau_{s-1}}{\omega}\right\}, \left\{\frac{1}{\omega}\right\}.$$

In particular, we have

$$\Psi(\omega) = \max_{0\leqslant j\leqslant s} \Lambda\left(\left\{\frac{\tau_j}{\omega}\right\},\omega\right).$$

Proof. Suppose that $\Lambda(\tau,\omega)$ is not differentiable at $\tau=\tau^*\in[0,1)$. Since $\Lambda(\tau,\omega)$ is a finite sum of piecewise linear functions, there exists at least one pair $(j,\ell)\in[0,s]\times\mathbb{Z}$ satisfying $\tau^*\omega+\ell\omega=\tau_j$. This implies $\ell=[\tau_j/\omega]$ and $\tau^*=\{\tau_j/\omega\}$. $\square$

Since the breakpoints of $\tilde{f}_\xi$ are 0, ξ, $1-\xi$ and 1, it follows from Lemmas 15 and 16 that

$$\Psi(\omega) = \max\left(\Lambda(0,\omega), \Lambda\left(\left\{\frac{\xi}{\omega}\right\},\omega\right)\right).$$

We next show that $\Lambda(0,\omega)\leqslant\Lambda(\{\xi/\omega\},\omega)$. To show this it is not necessary to calculate these values exactly, but we give them in the following lemma for the later use.

Lemma 17. *For a trapezoidal function $f_\xi(x)$ we have*

$$\Lambda(0,\omega) = \frac{\omega}{2\xi}\left(\Theta\left(\frac{1}{\omega}\right)-\Theta\left(\frac{\xi}{\omega}\right)-\Theta\left(\frac{1-\xi}{\omega}\right)\right)+\frac{1-\xi}{\omega}$$

and

$$\Lambda\left(\left\{\frac{\xi}{\omega}\right\},\omega\right) = \frac{\omega}{2\xi}\left(\Theta\left(\frac{\xi}{\omega}\right)+\Theta\left(\frac{1-\xi}{\omega}\right)-\Theta\left(\frac{1-2\xi}{\omega}\right)\right)+\frac{1-\xi}{\omega},$$

where $\Theta(x)=\{x\}(1-\{x\})$ is an even periodic continuous function on $\mathbb{R}$.

Proof. We divide the sum

$$\Lambda\left(\left\{\frac{\xi}{\omega}\right\},\omega\right) = \sum_{\ell\in\mathbb{Z}} \tilde{f}_\xi\left(\left\{\frac{\xi}{\omega}\right\}\omega+\ell\omega\right)$$

into three parts Σ_1, Σ_2, Σ_3 according as $\{\xi/\omega\}\omega+\ell\omega$ belongs to $[0,\xi]$, $(\xi,1-\xi]$, $(1-\xi,1]$ respectively. For Σ_1 the integer ℓ runs from 0 to $\ell_1=[\xi/\omega]$; hence

$$\Sigma_1 = \frac{\omega}{\xi} \sum_{0 \leqslant \ell \leqslant \ell_1} \left(\left\{ \frac{\xi}{\omega} \right\} + \ell \right) = \frac{\xi}{2\omega} + \frac{1}{2} + \frac{\omega}{2\xi} \Theta\left(\frac{\xi}{\omega} \right).$$

For Σ_2 the integer ℓ runs from $\ell_1 + 1$ to $\ell_2 = \ell_1 + [(1-2\xi)/\omega]$; so we have

$$\Sigma_2 = \sum_{\ell_1 < \ell \leqslant \ell_2} 1 = \left[\frac{1-2\xi}{\omega} \right].$$

Finally for Σ_3 the integer ℓ runs from $\ell_2 + 1$ to $\ell_3 = \ell_1 + [(1-\omega)/\xi]$; thus,

$$\Sigma_3 = \frac{\omega}{\xi} \sum_{\ell_2 < \ell \leqslant \ell_3} \left(\frac{1}{\omega} - \left\{ \frac{\xi}{\omega} \right\} - \ell \right) = \frac{\xi}{2\omega} - \frac{1}{2} + \left\{ \frac{1-2\xi}{\omega} \right\} + \frac{\omega}{2\xi} \left(\Theta\left(\frac{1-\xi}{\omega} \right) - \Theta\left(\frac{1-2\xi}{\omega} \right) \right).$$

Therefore

$$\Sigma_1 + \Sigma_2 + \Sigma_3 = \frac{\omega}{2\xi} \left(\Theta\left(\frac{\xi}{\omega} \right) + \Theta\left(\frac{1-\xi}{\omega} \right) - \Theta\left(\frac{1-2\xi}{\omega} \right) \right) + \frac{1-\xi}{\omega},$$

as required. We can evaluate the value $\Lambda(0, \omega)$ similarly. $\square$

Using the inequality $\Theta(x) + \Theta(y) \geqslant \Theta(x+y)$ holding for any $x, y \in \mathbb{R}$, we have immediately the following

Corollary 18. $\Lambda(0, \omega) \leqslant (1-\xi)/\omega \leqslant \Lambda(\{\xi/\omega\}, \omega)$.

Recall that $(1-\xi)/\omega$ is the average of $\Lambda(\tau, \omega)$ over one period in τ for trapezoidal functions. From this corollary we conclude that $\Psi(\omega) = \Lambda(\{\xi/\omega\}, \omega)$. More precisely Lemma 17 implies that

$$\Lambda\left(\left\{ \frac{\xi}{\omega} \right\}, \omega \right) - \Lambda(0, \omega) = \frac{\omega}{\xi} \min\left(\left\| \frac{\xi}{\omega} \right\|, \left\| \frac{1-\xi}{\omega} \right\| \right), \tag{11}$$

because the equality

$$\min(\|x\|, \|y\|) = \Theta(x) + \Theta(y) - \frac{1}{2}\left(\Theta(x+y) + \Theta(x-y) \right)$$

holds for any $x, y \in \mathbb{R}$.

To investigate the local maximum of $\Lambda(\tau, \omega)$, we distinguish two cases, as follows.

Case (a): $\{\xi/\omega\} + \{(1-\xi)/\omega\} < 1$.

Since $\{1/\omega\} = \{\xi/\omega\} + \{(1-\xi)/\omega\}$, both $\{\xi/\omega\}$ and $\{(1-\xi)/\omega\}$ lie in the interval $[0, \{1/\omega\}]$. By Corollary 18 the graph of Λ in τ is trapezoidal on $[0, \{1/\omega\}]$ and flat on $[\{1/\omega\}, 1]$.

Case (b): $\{\xi/\omega\} + \{(1-\xi)/\omega\} \geqslant 1$.

We have $\{1/\omega\} = \{\xi/\omega\} + \{(1-\xi)/\omega\} - 1$; so both $\{\xi/\omega\}$ and $\{(1-\xi)/\omega\}$ lie in the interval $(\{1/\omega\}, 1)$. Hence the graph of Λ in τ is flat on $[0, \{1/\omega\}]$ and trapezoidal on $[\{1/\omega\}, 1]$.

The maximum of $\Lambda(\tau, \omega)$ on $0 \leqslant \tau \leqslant \{1/\omega\}$ is therefore equal to $\Lambda(\{\xi/\omega\}, \omega)$ in case (a) and equal to $\Lambda(0, \omega)$ in case (b). Since case (b) occurs if and only if $(1-\xi)/\omega < 1$ for $1/2 < \omega \leqslant 1$, it follows from (11) that

$$\Psi_0(\omega) = \begin{cases} 0 & \text{for } 0 < \omega \leqslant 1-\xi, \\ (\omega + \xi - 1)/\xi & \text{for } 1-\xi < \omega \leqslant 1. \end{cases}$$

Hence we get

$$\sigma = \int_0^1 \psi(\omega)\,d\omega - \int_0^1 \Psi_0(\omega)\,d\omega = \int_0^1 \left(\Lambda\left(\left\{ \frac{\xi}{\omega} \right\}, \omega \right) - (1-\xi)\left(\left[\frac{1}{\omega} \right] + 1 \right) \right) d\omega - \frac{\xi}{2}$$

and, using Lemma 17,

$$\sigma = \frac{1}{2\xi}\big(Y(\xi) + Y(1-\xi) - Y(1-2\xi)\big) - (1-\xi)\gamma - \frac{\xi}{2},$$

where γ is Euler's constant and

$$Y(u) = \int_0^1 x\Theta\left(\frac{u}{x}\right) dx.$$

This integral can be easily evaluated as follows.

Lemma 19. *For* $0 < u \leqslant 1$ *we have* $Y(u) = u + \gamma u^2 + 2Z(u)$.

Proof. Substituting $y = x/u$ we have

$$u^{-2}Y(u) = \int_0^{1/u} y\Theta\left(\frac{1}{y}\right) dy = \int_0^1 y\Theta\left(\frac{1}{y}\right) dy + \frac{1}{u} - 1 + \log u.$$

The last integral is equal to

$$\begin{aligned}
\int_1^\infty \frac{\{x\}(1-\{x\})}{x^3}\,dx &= \sum_{\ell=1}^\infty \int_\ell^{\ell+1} \frac{(x-\ell)(\ell+1-x)}{x^3}\,dx \\
&= \sum_{\ell=1}^\infty \left(\log \ell - \log(\ell+1) + \frac{1}{2\ell} + \frac{1}{2(\ell+1)} \right) \\
&= \lim_{N\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{N+1} - \log(N+1) - \frac{1}{2} \right) \\
&= \gamma - \frac{1}{2},
\end{aligned}$$

which completes the proof. □

We thus conclude that

$$\sigma = 1 - \frac{\xi}{2} + \frac{1}{\xi}\big(Z(\xi) + Z(1-\xi) - Z(1-2\xi)\big) = 1 - \frac{\xi}{2} + \Delta_\xi Z(\xi),$$

because $Z(x)$ is an even function.

For the proof of Theorem 1 we take $\xi = 0.284$ and $\rho = 4.36$ so that

$$\beta = \Delta_\xi Z(\rho) - \rho \Delta_\xi Z'(\rho) = 0.15443703\ldots$$

and

$$\sigma = 1 - \frac{\xi}{2} + \Delta_\xi Z(\xi) = -0.42049786\ldots,$$

which clearly satisfy the condition $\beta > \sigma$. We also have

$$\kappa = \Delta_\xi Z'(\rho) = 0.18611205\ldots.$$

Finally the maximum of $W_\xi(u) = u\Delta_\xi Z'(\rho) - \Delta_\xi Z(u)$ on $[0,1]$ is attained at $u^* = 0.53137915\ldots$ and $\delta = W_\xi(u^*) = 1.5283879\ldots.$ Note that $u = u^*$ is a unique solution in $(0,1)$ of the equation $\Delta_\xi Z'(u) = \kappa$. Therefore

$$\Omega(f_\xi) = \frac{\beta + \delta}{\kappa(\beta - \sigma)} = 15.726995\ldots,$$

which completes the proof.

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